



Chapter 4 : Equivalence Relations

Lecture :

1. Relations and their properties
2. Representing Relations
3. Proprieties Relations
4. Combining Relations

Prepared by:

- Dr. Abbas Rammal
- Dr. Rabih Assaf

1 Relations and Their Properties

DEFINITION 1 Let A and B be sets. A *binary relation from A to B* is a subset of $A \times B$.

In other words, a binary relation from A to B is a set R of ordered pairs where the first element of each ordered pair comes from A and the second element comes from B . We use the notation $a R b$ to denote that $(a, b) \in R$ and $a \not R b$ to denote that $(a, b) \notin R$. Moreover, when (a, b) belongs to R , a is said to be **related to** b by R .

Functions as Relations

Recall that a function f from a set A to a set B (as defined in Section 2.3) assigns exactly one element of B to each element of A . The graph of f is the set of ordered pairs (a, b) such that $b = f(a)$. Because the graph of f is a subset of $A \times B$, it is a relation from A to B .

Relations on a Set

DEFINITION 2

A *relation on a set* A is a relation from A to A .

In other words, a relation on a set A is a subset of $A \times A$.

EXAMPLE 4

Let A be the set $\{1, 2, 3, 4\}$. Which ordered pairs are in the relation $R = \{(a, b) \mid a \text{ divides } b\}$?

Solution: Because (a, b) is in R if and only if a and b are positive integers not exceeding 4 such that a divides b , we see that

$$R = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 4), (3, 3), (4, 4)\}.$$

The pairs in this relation are displayed both graphically and in tabular form in Figure 2. 

Relations on a Set

EXAMPLE 5 Consider these relations on the set of integers:

$$R_1 = \{(a, b) \mid a \leq b\},$$

$$R_2 = \{(a, b) \mid a > b\},$$

$$R_3 = \{(a, b) \mid a = b \text{ or } a = -b\},$$


$$R_4 = \{(a, b) \mid a = b\},$$

$$R_5 = \{(a, b) \mid a = b + 1\},$$

$$R_6 = \{(a, b) \mid a + b \leq 3\}.$$

Which of these relations contain each of the pairs $(1, 1)$, $(1, 2)$, $(2, 1)$, $(1, -1)$, and $(2, 2)$?

Remark: Unlike the relations in Examples 1–4, these are relations on an infinite set.

Solution: The pair $(1, 1)$ is in R_1 , R_3 , R_4 , and R_6 ; $(1, 2)$ is in R_1 and R_6 ; $(2, 1)$ is in R_2 , R_5 , and R_6 ; $(1, -1)$ is in R_2 , R_3 , and R_6 ; and finally, $(2, 2)$ is in R_1 , R_3 , and R_4 . 

It is not hard to determine the number of relations on a finite set, because a relation on a set A is simply a subset of $A \times A$.

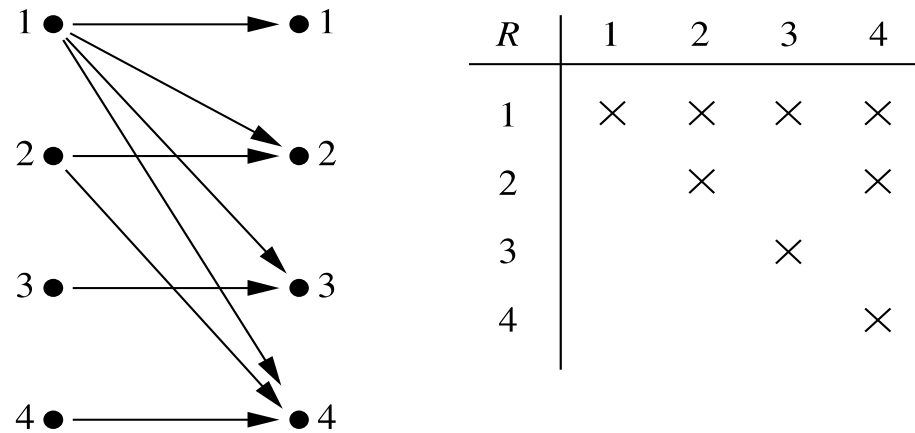


FIGURE 2 Displaying the Ordered Pairs in the Relation R from Example 4.

EXAMPLE 6 How many relations are there on a set with n elements?

Solution: A relation on a set A is a subset of $A \times A$. Because $A \times A$ has n^2 elements when A has n elements, and a set with m elements has 2^m subsets, there are 2^{n^2} subsets of $A \times A$. Thus, there are 2^{n^2} relations on a set with n elements. For example, there are $2^{3^2} = 2^9 = 512$ relations on the set $\{a, b, c\}$. ▶

Properties of Relations

There are several properties that are used to classify relations on a set. We will introduce the most important of these here.

In some relations an element is always related to itself. For instance, let R be the relation on the set of all people consisting of pairs (x, y) where x and y have the same mother and the same father. Then xRx for every person x .

DEFINITION 3

A relation R on a set A is called *reflexive* if $(a, a) \in R$ for every element $a \in A$.

Remark: Using quantifiers we see that the relation R on the set A is reflexive if $\forall a((a, a) \in R)$, where the universe of discourse is the set of all elements in A .

We see that a relation on A is reflexive if every element of A is related to itself. Examples 7–9 illustrate the concept of a reflexive relation.

EXAMPLE 7 Consider the following relations on $\{1, 2, 3, 4\}$:

$$R_1 = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 4), (4, 1), (4, 4)\},$$

$$R_2 = \{(1, 1), (1, 2), (2, 1)\},$$


$$R_3 = \{(1, 1), (1, 2), (1, 4), (2, 1), (2, 2), (3, 3), (4, 1), (4, 4)\},$$

$$R_4 = \{(2, 1), (3, 1), (3, 2), (4, 1), (4, 2), (4, 3)\},$$


$$R_5 = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 3), (3, 4), (4, 4)\},$$

$$R_6 = \{(3, 4)\}.$$

Which of these relations are reflexive?

Solution: The relations R_3 and R_5 are reflexive because they both contain all pairs of the form (a, a) , namely, $(1, 1)$, $(2, 2)$, $(3, 3)$, and $(4, 4)$. The other relations are not reflexive because they do not contain all of these ordered pairs. In particular, R_1 , R_2 , R_4 , and R_6 are not reflexive because $(3, 3)$ is not in any of these relations. 

EXAMPLE 9 Is the “divides” relation on the set of positive integers reflexive?

Solution: Because $a \mid a$ whenever a is a positive integer, the “divides” relation is reflexive. (Note that if we replace the set of positive integers with the set of all integers the relation is not reflexive because by definition 0 does not divide 0.) 

DEFINITION 4

A relation R on a set A is called *symmetric* if $(b, a) \in R$ whenever $(a, b) \in R$, for all $a, b \in A$. A relation R on a set A such that for all $a, b \in A$, if $(a, b) \in R$ and $(b, a) \in R$, then $a = b$ is called *antisymmetric*.

Remark: Using quantifiers, we see that the relation R on the set A is symmetric if $\forall a \forall b ((a, b) \in R \rightarrow (b, a) \in R)$. Similarly, the relation R on the set A is antisymmetric if $\forall a \forall b (((a, b) \in R \wedge (b, a) \in R) \rightarrow (a = b))$.

EXAMPLE 11

Which of the relations from Example 5 are symmetric and which are antisymmetric?

Solution: The relations R_3 , R_4 , and R_6 are symmetric. R_3 is symmetric, for if $a = b$ or $a = -b$, then $b = a$ or $b = -a$. R_4 is symmetric because $a = b$ implies that $b = a$. R_6 is symmetric because $a + b \leq 3$ implies that $b + a \leq 3$. The reader should verify that none of the other relations is symmetric.


The relations R_1 , R_2 , R_4 , and R_5 are antisymmetric. R_1 is antisymmetric because the inequalities $a \leq b$ and $b \leq a$ imply that $a = b$. R_2 is antisymmetric because it is impossible that $a > b$ and $b > a$. R_4 is antisymmetric, because two elements are related with respect to R_4 if and only if they are equal. R_5 is antisymmetric because it is impossible that $a = b + 1$ and $b = a + 1$. The reader should verify that none of the other relations is antisymmetric. ◀

DEFINITION 5


A relation R on a set A is called *transitive* if whenever $(a, b) \in R$ and $(b, c) \in R$, then $(a, c) \in R$, for all $a, b, c \in A$.

Remark: Using quantifiers we see that the relation R on a set A is transitive if we have $\forall a \forall b \forall c ((a, b) \in R \wedge (b, c) \in R) \rightarrow (a, c) \in R$.

EXAMPLE 14 Which of the relations in Example 5 are transitive?

Solution: The relations R_1, R_2, R_3 , and R_4 are transitive. R_1 is transitive because $a \leq b$ and $b \leq c$ imply that $a \leq c$. R_2 is transitive because $a > b$ and $b > c$ imply that $a > c$. R_3 is transitive because $a = \pm b$ and $b = \pm c$ imply that $a = \pm c$. R_4 is clearly transitive, as the reader should verify. R_5 is not transitive because $(2, 1)$ and $(1, 0)$ belong to R_5 , but $(2, 0)$ does not. R_6 is not transitive because $(2, 1)$ and $(1, 2)$ belong to R_6 , but $(2, 2)$ does not. 

EXAMPLE 15 Is the “divides” relation on the set of positive integers transitive?

Solution: Suppose that a divides b and b divides c . Then there are positive integers k and l such that $b = ak$ and $c = bl$. Hence, $c = a(kl)$, so a divides c . It follows that this relation is transitive. 

Combining Relations

Because relations from A to B are subsets of $A \times B$, two relations from A to B can be combined in any way two sets can be combined. Consider Examples 17–19.

EXAMPLE 17 Let $A = \{1, 2, 3\}$ and $B = \{1, 2, 3, 4\}$. The relations $R_1 = \{(1, 1), (2, 2), (3, 3)\}$ and $R_2 = \{(1, 1), (1, 2), (1, 3), (1, 4)\}$ can be combined to obtain

$$R_1 \cup R_2 = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (3, 3)\},$$

$$R_1 \cap R_2 = \{(1, 1)\},$$


$$R_1 - R_2 = \{(2, 2), (3, 3)\},$$

$$R_2 - R_1 = \{(1, 2), (1, 3), (1, 4)\}.$$



EXAMPLE 19 Let R_1 be the “less than” relation on the set of real numbers and let R_2 be the “greater than” relation on the set of real numbers, that is, $R_1 = \{(x, y) \mid x < y\}$ and $R_2 = \{(x, y) \mid x > y\}$. What are $R_1 \cup R_2$, $R_1 \cap R_2$, $R_1 - R_2$, $R_2 - R_1$, and $R_1 \oplus R_2$?

Solution: We note that $(x, y) \in R_1 \cup R_2$ if and only if $(x, y) \in R_1$ or $(x, y) \in R_2$. Hence, $(x, y) \in R_1 \cup R_2$ if and only if $x < y$ or $x > y$. Because the condition $x < y$ or $x > y$ is the same as the condition $x \neq y$, it follows that $R_1 \cup R_2 = \{(x, y) \mid x \neq y\}$. In other words, the union of the “less than” relation and the “greater than” relation is the “not equals” relation.

Next, note that it is impossible for a pair (x, y) to belong to both R_1 and R_2 because it is impossible that $x < y$ and $x > y$. It follows that $R_1 \cap R_2 = \emptyset$. We also see that $R_1 - R_2 = R_1$, $R_2 - R_1 = R_2$, and $R_1 \oplus R_2 = R_1 \cup R_2 - R_1 \cap R_2 = \{(x, y) \mid x \neq y\}$. 

DEFINITION 6

Let R be a relation from a set A to a set B and S a relation from B to a set C . The *composite* of R and S is the relation consisting of ordered pairs (a, c) , where $a \in A$, $c \in C$, and for which there exists an element $b \in B$ such that $(a, b) \in R$ and $(b, c) \in S$. We denote the composite of R and S by $S \circ R$.

EXAMPLE 20 What is the composite of the relations R and S , where R is the relation from $\{1, 2, 3\}$ to $\{1, 2, 3, 4\}$ with $R = \{(1, 1), (1, 4), (2, 3), (3, 1), (3, 4)\}$ and S is the relation from $\{1, 2, 3, 4\}$ to $\{0, 1, 2\}$ with $S = \{(1, 0), (2, 0), (3, 1), (3, 2), (4, 1)\}$?

Solution: $S \circ R$ is constructed using all ordered pairs in R and ordered pairs in S , where the second element of the ordered pair in R agrees with the first element of the ordered pair in S . For example, the ordered pairs $(2, 3)$ in R and $(3, 1)$ in S produce the ordered pair $(2, 1)$ in $S \circ R$. Computing all the ordered pairs in the composite, we find

$$S \circ R = \{(1, 0), (1, 1), (2, 1), (2, 2), (3, 0), (3, 1)\}.$$


DEFINITION 7

Let R be a relation on the set A . The powers R^n , $n = 1, 2, 3, \dots$, are defined recursively by

$$R^1 = R \quad \text{and} \quad R^{n+1} = R^n \circ R.$$

The definition shows that $R^2 = R \circ R$, $R^3 = R^2 \circ R = (R \circ R) \circ R$, and so on.

EXAMPLE 22 Let $R = \{(1, 1), (2, 1), (3, 2), (4, 3)\}$. Find the powers R^n , $n = 2, 3, 4, \dots$.

Solution: Because $R^2 = R \circ R$, we find that $R^2 = \{(1, 1), (2, 1), (3, 1), (4, 2)\}$. Furthermore, because $R^3 = R^2 \circ R$, $R^3 = \{(1, 1), (2, 1), (3, 1), (4, 1)\}$. Additional computation shows that R^4 is the same as R^3 , so $R^4 = \{(1, 1), (2, 1), (3, 1), (4, 1)\}$. It also follows that $R^n = R^3$ for $n = 5, 6, 7, \dots$. The reader should verify this. 

THEOREM 1 The relation R on a set A is transitive if and only if $R^n \subseteq R$ for $n = 1, 2, 3, \dots$